

The Potts Model Built on Sand

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We consider the $q=4$ Potts model on the square lattice with an additional non-local interaction. That interaction arises from the choice of the reference measure taken to be the uniform measure on the recurrent configurations for the abelian sandpile model. In that reference measure some correlation functions have a power-law decay. We investigate the low-temperature phase diagram and we prove the existence of a single stable phase with exponential decay of correlations. For all boundary conditions the density of 4 in the infinite volume limit goes to one as the temperature tends to zero.

KEY WORDS: Sandpile model; Potts interaction; nonlocal constraints.

1. INTRODUCTION

Over the last 15 years the concept of self-organized criticality has become very popular in science. While controversy remains in how new or original the idea is, certainly a whole literature has been inspired by the concept and a multitude of models and phenomena have been investigated in that light. It is indeed remarkable to observe such a universality in power-law response behavior for a wide variety of phenomena and over many different scales. One of the main models or even paradigms where self-organized criticality has been studied is the abelian sandpile model introduced in ref. 1. The model is attractive not only because it is simple to define and to simulate but also because it has an interesting mathematical

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structure. The sandpile model gives rise to an elegant algebraic formalism and some aspects of the stationary measure are exactly solvable, see ref. 2. We will remind the reader of the sandpile model in Section 2.2.

One of the important questions is to understand the robustness of the phenomenon of self-organized criticality. Specifically, what remains of the criticality when adding an interaction to the sandpile model. Obviously, there are trivial ways to destroy the critical behavior. For example, in two dimensions, we could add a sufficiently large chemical potential to stabilize the 3-phase or the 4-phase. Here we choose to add an interaction that is symmetric with respect to the possible heights. Another input could certainly be that in close to physical realizations of the sandpile model, the grains at neighboring sites could stick together, preferring neighboring heights at the same level. The simplest realization is then the Potts interaction. The first question is: what phase will survive for positive or even sufficiently low temperatures. The present paper gives an answer: the maximal height wins (4's dominate over 3's in two dimensions). At the same time, we prove exponential decay of the correlations in the corresponding low temperature phase.

From a different angle, our model adds a rather unusual *a priori* or reference measure to the standard Potts model. The reference measure, being concentrated on the stationary measure of the sandpile model, has nonlocal features and introduces constraints. We are not aware of previous studies of Gibbs measures in such an environment and indeed many of the first results (such as existence of thermodynamic limit) and of the standard techniques (such as the usual cluster expansion) break down or have no immediate answer when *a priori* measure possesses such nonlocal constraints. In the paper we observe that the method of random cluster representation can be stretched to include the study of our Potts model built on sand. We use it to show the stability of the maximal phase. To prove the domination of that maximal phase over the others we use some extended contour ensemble method. It can be seen as an extension of the Dinaburg-Sinai techniques of ref. 3.

Open questions remain. The domination results are here restricted to low temperature. Most interesting is the question whether there would be a cross-over from exponential to power-law like behavior of correlations at some finite temperature or whether the self-organized criticality disappears at arbitrarily high temperatures.

We start in the next section with the details of our model. The results are collected in Section 3. Section 4 proves the stability of the maximal phase and the last section contains the cluster expansion formalism that proves the instability of the other phases.

2. MODEL

We define the model on the two-dimensional lattice \mathbf{Z}^2 . Lattice sites are denoted by x, y, z and we write $x \sim y$ if x and y are nearest neighbors. For a subset $V \subset \mathbf{Z}^2$ we denote by ∂V the exterior boundary of V , i.e., the set of those $y \in \mathbf{Z}^2 \setminus V$ such that there exists a nearest neighbor of y in V , $\bar{V} = V \cup \partial V$, and the set $\partial(\mathbf{Z}^2 \setminus V)$ is called the inner boundary of V . The square $[-n, n]^2 \cap \mathbf{Z}^2$ is denoted by V_n . The configuration space is $\Omega = \{1, 2, 3, 4\}^{\mathbf{Z}^2}$. Elements of Ω are written as η, ξ . For a configuration η , $\eta(x) \in \{1, 2, 3, 4\}$ is interpreted as the “number of sand grains” at x . For $n \in \mathbb{N}$, $\Omega_n = \{1, 2, 3, 4\}^{V_n}$ denotes the set of finite volume height configurations. Below we introduce measures $\mu_{\beta,n}^a$ on Ω_n that correspond to the finite volume Potts model at inverse temperature β with boundary condition $a \in \{1, 2, 3, 4\}$, restricted to a special set of “recurrent configurations” defined from the abelian sandpile model (cf. Section 2.2). Our main result is that for β large, $\mu_{\beta,n}^a$ forms the single stable phase of that model.

2.1. Potts Model

The Potts Hamiltonian with fixed boundary condition $a \in \{1, 2, 3, 4\}$ on the volume V_n is

$$H_n(\eta|a) = \sum_{x \sim y \in V_n} I[\eta(x) \neq \eta(y)].$$

That is a finite sum over nearest-neighbor pairs of sites of which at least one belongs to V_n and where it is understood that we substitute $\eta(z) = a$ whenever $z \notin V_n$.

The Hamiltonians $H_n(\eta|a)$ give rise to the finite volume Gibbs measures on Ω_n

$$\nu_{\beta,n}^a(\eta) = \frac{1}{Z_{\beta,n}^a} \exp[-\beta H_n(\eta|a)],$$

where $\beta > 0$ is the inverse temperature and where the normalizing factor $Z_{\beta,n}^a$ is the partition function. It is well-known that there exists a critical inverse temperature $\beta_c \in (0, +\infty)$ such that for $\beta < \beta_c$ the Potts model has a unique infinite volume Gibbs measure (as $n \uparrow +\infty$), i.e., the weak limits of $\nu_{\beta,n}^a$ for different a coincide, while for $\beta > \beta_c$, these weak limits are all different and define four mutually singular ergodic Gibbs measures ν_β^a on Ω , called the pure phases, see for example ref. 4.

2.2. Sandpile Model

The abelian sandpile model in volume V_n is a Markov chain on Ω_n . We briefly introduce that Markov chain, more details can be found in the original paper^(1,2,5). The Markov chain starting from $\eta_0 \in \Omega_n$ is defined as follows. Suppose that η_{t-1} is the configuration at time $t-1 \geq 0$. Pick randomly a site in V_n , say $x \in V_n$, and add one grain at x to η_{t-1} . In case $\eta_{t-1}(x) \leq 3$, the new configuration is simply

$$\eta_t(y) = \eta_{t-1}(y) + \delta_{x,y}$$

with $\delta_{x,y}$ the Kronecker delta. In case $\eta_{t-1}(x) = 4$, by adding one grain at x the number of grains at x becomes equal to 5. That site will now topple, i.e., 4 grains are removed from x and one grain is given to each neighbor of x in V_n . At the boundary, grains are lost when the site topples. It is now possible that the number of grains at one or more neighbors of x exceeds 4 and we have to repeat the toppling operation on all of these, and so on. It turns out that no matter in what order we perform these toppling operations, at the end of the avalanche a unique configuration $\eta_t \in \Omega_n$ appears. In that way, a discrete time Markov chain on Ω_n is defined where the only randomness is in the independently repeated uniform choice of the site where a grain is added.

Analysis of that Markov chain shows that it has a unique class \mathcal{R}_n of recurrent configurations and the stationary measure λ_n is uniform on that class

$$\lambda_n(\eta) = \frac{1}{|\mathcal{R}_n|} I[\eta \in \mathcal{R}_n],$$

see ref. 2.

Whether a particular configuration $\eta \in \Omega_n$ belongs to \mathcal{R}_n can be decided from the output of the so-called burning algorithm⁽²⁾. The burning algorithm has as an input the configuration η and its output is a set $A \subset V_n$. It runs as follows: start from $A_0 = V_n$ and remove (“burn”) all those vertices $x \in A_0$ (and edges containing x) which satisfy $\eta(x) > n_{A_0}(x)$ where $n_V(x)$ denotes the number of neighbors of x in V . This gives A_1 ; now proceed in the same way with A_1 , etc. until no further vertices can be removed. The output A of the algorithm is the set of remaining vertices. Recurrence is then characterized by “burnability”, i.e., $\eta \in \mathcal{R}_n$ if and only if $A = \emptyset$, i.e., all vertices can be burned.

The stationary measure λ_n is thus the uniform probability measure on all burnable configurations in V_n . The cardinality $|\mathcal{R}_V|$ (=the number of recurrent configurations in V) equals the determinant of the discrete

Laplacian on V with open boundary conditions see ref.2. E.g., if V is a square containing N sites, then $|\mathcal{R}_V| \simeq (3.21)^N$.

For a proof of these facts, see e.g., refs. 2,5–7. Remark that for all finite $V \subset \mathbb{Z}^2$, the constant configurations $\eta \equiv 4$ and $\eta \equiv 3$ are in \mathcal{R}_V , but $\eta \equiv 2$ and $\eta \equiv 1$ are not recurrent except for some very special choices of V . One easily concludes that the condition that $\eta \in \mathcal{R}_n$ is a nonlocal constraint.

The following proposition is an immediate consequence of the burning algorithm.

Proposition 2.1. If $\eta \in \mathcal{R}_n$ and $\xi \geq \eta$ (pointwise), then $\xi \in \mathcal{R}_n$.

In some aspects the abelian sandpile measure $\lambda_n, n \rightarrow \infty$, behaves as a model of statistical mechanics at the critical point, a phenomenon which is sometimes referred to as “self-organized criticality” because there is no explicit tuning of parameters. In the physics literature various critical exponents related to the avalanche behavior are introduced for that model. One signature of “critical behavior” is the presence of power law decay of correlations for the height 1 two-point function, as proven by Majumdar and Dhar in ref. 8.

Proposition 2.2. There exist constants $c, C > 0$ such that

$$c|x|^{-4} \leq |\lambda_n(\eta(0) = \eta(x) = 1) - \lambda_n(\eta(0) = 1)\lambda_n(\eta(x) = 1)| \leq C|x|^{-4}$$

for all $x \neq 0$ and n large enough.

On the other hand, a contour of 4’s completely decouples the inside and the outside, as we now show. A subset $V \subset \mathbb{Z}^2$ is called simply connected if the corresponding $\hat{V} \subset \mathbb{R}^2$ obtained by “filling the squares” of V is simply connected.

Proposition 2.3. For $W \subset V_n$ denote by 4_W the event that $\eta(x) = 4$ on W . For any simply connected subset $V \subset \mathbb{Z}^2$ with $\bar{V} \subset V_n$

$$\lambda_n(\eta_V \eta_{V_n \setminus \bar{V}} | 4_{\partial V}) = \lambda_n(\eta_V | 4_{\partial V}) \lambda_n(\eta_{V_n \setminus \bar{V}} | 4_{\partial V}). \tag{2.1}$$

Proof. Denote by $\mathcal{R}_{V_n \setminus \bar{V}}^{\text{ext}}$ the set of configurations which are burnable in $V_n \setminus \bar{V}$ and such that the extension $\eta_{V_n \setminus \bar{V}} 4_{\bar{V}}$ is burnable in V_n . By the burning algorithm, $\eta_V 4_{\partial V} \eta_{V_n \setminus \bar{V}} \in \mathcal{R}_{V_n}$ if and only if $\eta_V \in \mathcal{R}_V$ and $\eta_{V_n \setminus \bar{V}} \in \mathcal{R}_{V_n \setminus \bar{V}}^{\text{ext}}$. Therefore

$$\lambda_n(\eta_V \eta_{V_n \setminus \bar{V}} | 4_{\partial V}) = \frac{I[\eta_V \in \mathcal{R}_V, \eta_{V_n \setminus \bar{V}} \in \mathcal{R}_{V_n \setminus \bar{V}}^{\text{ext}}]}{|\mathcal{R}_{V_n \setminus \bar{V}}^{\text{ext}}| |\mathcal{R}_V|}, \quad (2.2)$$

$$\lambda_n(\eta_V | 4_{\partial V}) = \frac{I[\eta_V \in \mathcal{R}_V]}{|\mathcal{R}_V|}, \quad (2.3)$$

$$\lambda_n(\eta_{V_n \setminus \bar{V}} | 4_{\partial V}) = \frac{I[\eta_{V_n \setminus \bar{V}} \in \mathcal{R}_{V_n \setminus \bar{V}}^{\text{ext}}] |\mathcal{R}_V|}{|\mathcal{R}_{V_n \setminus \bar{V}}^{\text{ext}}| |\mathcal{R}_V|}, \quad (2.4)$$

which gives the result. \blacksquare

2.3. Sandpile Model with Potts Interaction

Define the probability measures $\mu_{\beta,n}^a$ on Ω_n as

$$\mu_{\beta,n}^a(\eta) = \frac{\exp(-\beta H_n(\eta|a)) I[\eta \in \mathcal{R}_n]}{\Xi_{\beta,n}^a},$$

where the normalizing constant $\Xi_{\beta,n}^a$ is

$$\Xi_{\beta,n}^a \equiv \sum_{\eta \in \mathcal{R}_n} \exp(-\beta H_n(\eta|a)).$$

Similarly, we define the partition function $\Xi_{\beta,V}^a$ in an arbitrary finite volume V . This partition function will also be abbreviated as $\Xi_{a,V}$. $\mu_{\beta,n}^a$ is of course just the original Potts measure conditioned on being recurrent:

$$\mu_{\beta,n}^a(\eta) = \nu_{\beta,n}^a(\eta | \mathcal{R}_n).$$

Obviously, at infinite temperature, $\beta=0$, we recover the stationary measure λ_n of the sandpile model. The constraint $\eta \in \mathcal{R}_n$ can be viewed as introducing an extra nonlocal interaction (implicitly given by the burning algorithm) but it also breaks the Potts-symmetry: approximately for $n \uparrow +\infty$, $\lambda_n(\eta(0)=4) = 0.4$, $\lambda_n(\eta(0)=3) = 0.3$, $\lambda_n(\eta(0)=2) = 0.2$, $\lambda_n(\eta(0)=1) = 0.1$ as computed by Priezzhev.⁽⁹⁾

3. RESULTS

With boundary condition $a=4$, at low temperature, the typical configurations of the Potts model on sand look like an ocean of 4's with exponentially rare burnable islands.

Theorem 3.1. For any $\epsilon > 0$ there exists $\beta_0 \in (0, \infty)$ such that for all $\beta > \beta_0$ and all $n \in \mathbb{N}$

$$\mu_{\beta,n}^4(\eta(0)=4) > 1 - \epsilon. \tag{3.1}$$

Moreover there exists $c > 0$ such that for $\beta > \beta_0$ and n big enough we have the bound

$$|\mu_{\beta,n}^4(\eta(x)\eta(0)) - \mu_{\beta,n}^4(\eta(x))\mu_{\beta,n}^4(\eta(0))| \leq e^{-c|x|} \tag{3.2}$$

exponentially small in the distance $|x|$ from the origin.

Equation (3.2) must be contrasted with the situation for $\beta = 0$ where there are long range correlations, (see Proposition 2.2).

Besides “all 4”, the “all 3” is the only other groundstate. But that one is unstable.

Theorem 3.2. For every $\alpha > 0$ there exists $\beta(\alpha) \in (0, \infty)$ and $c = c(\alpha, \beta) > 0$ such that for all $\beta > \beta(\alpha)$

$$\mu_{\beta,n}^3(|\{x \in V_n : \eta(x) = 3\}| > \alpha |V_n|) \leq e^{-c|V_n|} \tag{3.3}$$

for large n .

Section 4 gives the proof of Theorem 3.1 and introduces a random cluster representation of the Potts model on sand. Section 5 is devoted to the proof of Theorem 3.2. It will be seen that, as an extension of Theorem 3.2, for no matter what boundary conditions, the density of 4 tends to one with $\beta \uparrow +\infty$.

4. RANDOM CLUSTER REPRESENTATION

The volume $\overline{V}_n = V_n \cup \partial V_n$ can be considered as a finite graph with the sites $x \in V_n \cup \partial V_n$ as vertices and with edge set $B_n = B$ consisting of the nearest neighbor bonds $x \sim y$ where at least one neighbor is in V_n . We define the *sand-Potts* random cluster measure $\varphi_{p,n}^a = \varphi_p^a$ on this graph with parameter $p \in [0, 1]$ as the probability measure on $\{0, 1\}^B$ which to each $\sigma \in \{0, 1\}^B$ assigns probability

$$\varphi_p^a(\sigma) = \frac{1}{N_p^a} \left[\prod_{e \in B} p^{\sigma(e)} (1-p)^{1-\sigma(e)} \right] \sum_{\eta \in \mathcal{R}_n} I, [\eta \text{ is constant on clusters of } \sigma.] \tag{4.1}$$

By cluster we mean a (nearest-neighbor) connected component of sites (including isolated sites) as obtained from the bond configuration σ . Bonds for which $\sigma(e) = 1$, $\sigma(e) = 0$ are called open, respectively, closed. In this definition, we assume that the boundary sites are all connected (wired). All sites that are connected to the boundary are in the same cluster. The restriction that η is constant on clusters also implies that η is constant equal to a on the cluster of the boundary. Remember however that $\eta \equiv 2$ and $\eta \equiv 1$ are not in \mathcal{R}_n .

4.1. Stochastic Domination

Lemma 4.1. Let $a = 3$ or $a = 4$. For every edge $e = \langle xy \rangle$ in B and every $\sigma_{B \setminus e} \in \{0, 1\}^{B \setminus \{e\}}$,

$$\varphi_p^a(\sigma(e) = 1 | \sigma_{B \setminus e}) = p \tag{4.2}$$

if x and y are connected via open edges in $\sigma_{B \setminus e}$. If, on the other hand, x and y are not connected via open edges in $\sigma_{B \setminus e}$, then we still have

$$p \geq \varphi_p^a(\sigma(e) = 1 | \sigma_{B \setminus e}) \tag{4.3}$$

and for $a = 4$,

$$\varphi_p^4(\sigma(e) = 1 | \sigma_{B \setminus e}) \geq \frac{p}{7 - 6p}. \tag{4.4}$$

Proof. Let $\sigma \in \{0, 1\}^B$. We write

$$\sum_{\eta \in \mathcal{R}_n} I[\eta \text{ is constant on clusters}] = k(n, a; \sigma) \tag{4.5}$$

for the number of recurrent configurations that are constant on the σ -clusters and fixed equal to a for each site that is σ -connected to ∂V_n . It equals $|\mathcal{R}_n|$ when all edges in σ are closed. Obviously, $k(n, a; \sigma) \leq |\mathcal{R}_n|$ and $k(n, a; \sigma)$ is decreasing in σ and is increasing in a . For $a = 3$ or 4 , $k(n, a; \sigma) \geq 1$. Continuing with either $a = 3$ or $a = 4$, we have

$$\frac{\varphi_p^a(1_e \sigma_{B \setminus e})}{\varphi_p^a(0_e \sigma_{B \setminus e})} = \frac{p k(n, a; 1_e \sigma_{B \setminus e})}{(1 - p) k(n, a; 0_e \sigma_{B \setminus e})}, \tag{4.6}$$

and hence

$$\varphi_p^a(\sigma(e) = 1 | \sigma_{B \setminus e}) = \frac{1}{1 + \frac{\varphi_p^a(0_e \sigma_{B \setminus e})}{\varphi_p^a(1_e \sigma_{B \setminus e})}} = \frac{p}{p + (1-p) \frac{k(n, a; 0_e \sigma_{B \setminus e})}{k(n, a; 1_e \sigma_{B \setminus e})}}. \tag{4.7}$$

Abbreviate $\sigma^{0,e} = 0_e \sigma_{B \setminus e}$ and $\sigma^{1,e} = 1_e \sigma_{B \setminus e}$; they are both equal to $\sigma_{B \setminus e}$ off e and $\sigma^{0,e}(e) = 0$ and $\sigma^{1,e}(e) = 1$.

To prove the first statement (4.2): suppose x, y are connected via open edges in $\sigma_{B \setminus e}$, then every configuration $\eta \in \mathcal{R}_n$ compatible with σ has $\eta(x) = \eta(y)$, and hence $k(n, a; \sigma)$ does not depend on $\sigma(e)$ in that case. However, if x and y are not connected via open edges in $\sigma_{B \setminus e}$, then we must investigate the effect of merging two clusters. By making e open, we connect two clusters and we must estimate the new number of recurrent configurations that are constant on clusters in terms of the old. Since always

$$k(n, a; \sigma^{0,e}) \geq k(n, a; \sigma^{1,e}), \tag{4.8}$$

we obtain (4.3) from (4.7). For the last statement (4.4), we combine Proposition 2.1 with (4.7). Suppose that $\eta \in \mathcal{R}_n$ and is constant on clusters C_1 and C_2 taking there the values a_1 and a_2 , respectively. The new configuration ξ defined as

$$\xi(x) = \eta(x), \quad x \notin C_1 \cup C_2, \quad \xi(x) = \max\{a_1, a_2\}, \quad x \in C_1 \cup C_2$$

is still recurrent and is constant on $C_1 \cup C_2$. Moreover, if say C_1 is the boundary cluster, then necessarily $a_1 = 4$ and hence also $\max\{a_1, a_2\} = 4$ remains compatible with the boundary. (This does not work with the boundary condition $a = 3$.) Simple counting shows that the map $\eta \rightarrow \xi$ is at most seven to one, or

$$k(n, 4; \sigma^{0,e}) \leq 7k(n, 4; \sigma^{1,e}), \tag{4.9}$$

combination of (4.9) and (4.7) gives (4.4). ■

Let ψ_q be the Bernoulli product measure on $\{0, 1\}^B$ with density $q = \psi_q(\sigma_e = 1)$.

Proposition 4.2. The random cluster measure φ_p^4 stochastically dominates ψ_q with $q = p/(7 - 6p)$, i.e., $\varphi_p^4(\sigma(e) = 1, e \in E) \geq \psi_q(\sigma(e) = 1, e \in E)$

for all edge sets E . For $a = 3, 4$, φ_p^a is stochastically dominated by ψ_p . Finally, φ_p^a always stochastically dominates $\varphi_{p'}^a$ for $0 \leq p' \leq p \leq 1$.

Proof. The first and second statement follows directly from Lemma 4.1, see e.g., Theorem 4.8 in ref. 4. The last statement follows from (4.7) and (4.8). ■

4.2. Coupling

The previous construction and arguments are analogous to and inspired by the Fortuin–Kasteleyn representation of the standard Potts model. To recover the q -state Potts model, one should simply replace \mathcal{R}_n in (4.5) with Ω_n . Our next step, making a coupling between the η - and the σ -field, is the analogue of the Swendsen–Wang–Edwards–Sokal coupling.^(10,11) For a general reference, see ref. 4.

We make a coupling $\mathbb{P}_{p,n}^a = \mathbb{P}_p^a$ between the Potts model on sand and the sand-Potts random cluster measure. Let \mathbb{P}_p^a be the probability measure on $\Omega_n \times \{0, 1\}^B$ constructed as follows. Assign first to each site in V_n a sandvalue according to the probability measure λ_n and each site at the boundary ∂V_n gets the value a . Independently, let each edge in B take the value 0 or 1 with probabilities $1 - p$ and p , respectively. Secondly, condition on the event that no two neighboring sites (including sites at the boundary) with different heights have an open edge connecting them. In a formula,

$$\mathbb{P}_p^a(\eta, \sigma) = \frac{1}{M_p^a} I[\eta \in \mathcal{R}_n] \prod_{e=(xy) \in B} [p^{\sigma(e)}(1-p)^{1-\sigma(e)} I[(\eta_x - \eta_y)\sigma(e) = 0]],$$

where in the last indicator function it is understood that $\eta(z) = a$ for $z \in \partial V_n$.

Proposition 4.3. Suppose $\beta = -\ln(1 - p)$. Then, $\mu_{\beta,n}^a$ is the marginal of \mathbb{P}_p^a projected on Ω_n and φ_p^a is the marginal of \mathbb{P}_p^a projected on $\{0, 1\}^B$.

Proof. The proof is by direct computation. For example, if we sum over the σ we have

$$\sum_{\sigma} \prod_{e \in B} [p^{\sigma(e)}(1-p)^{1-\sigma(e)} I[(\eta(x) - \eta(y))\sigma(e) = 0]] = (1-p)^{I[\eta(x) \neq \eta(y)]}$$

which determines $1 - p = \exp[-\beta]$. ■

4.3. Proof of Theorem 3.1: Stability of the 4-Phase

We apply Proposition 4.3

$$\begin{aligned}
 & \mu_{\beta,n}^4(\eta(0)=4) \\
 & \geq \mathbb{P}_p^4(\eta(0)=4 \mid 0 \text{ is in the cluster of the boundary}) \\
 & \quad \times \varphi_p^4(0 \text{ is in the cluster of the boundary}) \\
 & = \varphi_p^4(0 \text{ is in the cluster of the boundary}), \tag{4.10}
 \end{aligned}$$

and now Proposition 4.2 to conclude that

$$\mu_{\beta,n}^4(\eta(0)=4) \geq \psi_{p/(7-6p)} \quad [0 \text{ is connected to the boundary}]$$

which goes to one, uniformly in n as $\beta = -\ln(1-p)$ goes to $+\infty$. That shows (3.1). For the exponential decay of correlations, (3.2), observe that by the very same argument as above one shows that for large β , in $\mu_{\beta,n}^4$ there is percolation of 4's uniformly in n . For each n the $\mu_{\beta,n}^4$ —probability that there is a nearest-neighbor path of 4's connecting the origin with the boundary ∂V_n is not smaller than the percolation probability in the Bernoulli bond process with occupation probability $(1 - \exp[-\beta]) / (1 + 6 \exp[-\beta])$. Moreover, we can always consider a rectangle parallel to V_n between the origin and site x with one side proportional to $|x|$ and the other side equal to n . Again by the same domination argument, the $\mu_{\beta,n}^4$ —probability that in that rectangle, there is percolation of 4's from one side of V_n to the opposite side is not smaller than $1 - \exp[-c|x|]$ with $c \uparrow +\infty$ as $\beta \uparrow +\infty$, uniformly in n . Therefore, denoting that event by “crossing”, we have that

$$|\mu_{\beta,n}^4(\eta(x)\eta(0)) - \mu_{\beta,n}^4(\eta(x)\eta(0)|\text{crossing})| \leq \exp[-c|x|].$$

That can be combined with Proposition 2.3; if the origin and the site x are separated by a path of 4's, then they are independent

$$\mu_{\beta,n}^4(\eta(x)\eta(0)|\text{crossing}) = \mu_{\beta,n}^4(\eta(x)|\text{crossing})\mu_{\beta,n}^4(\eta(0)|\text{crossing}).$$

For each of the two factors we can use the previous argument to conclude the proof.

5. INSTABILITY OF THE 3-PHASE

In this section we prove Theorem 3.2.

The main idea is to consider the restricted ensemble defined below (following ideas from ref. 3). Let V be a finite (large) volume and $\eta \in \Omega_V$ a configuration in the volume V . Define the contours of η as connected components of edges separating the different values of η in neighboring sites (we now consider these sites as centers of unit squares and the edges are the sides of these squares). We suppose that the configuration $\eta \in \mathcal{R}_V$ is burnable. The set of configurations that have contours with lengths not exceeding 12 not touching ∂V is denoted by \mathcal{R}_V^{12} . That set of configurations defines a restricted ensemble. It is easy to see that we have the following alternative: either the configuration η is a sea of 4 with islands made from contours with length ≤ 12 , or it is a sea of 3 with such islands

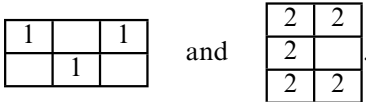
$$\mathcal{R}_V^{12} = \mathcal{R}_{V,4}^{12} \cup \mathcal{R}_{V,3}^{12}. \tag{5.1}$$

Notice that a sea of 2 or 1 is not possible by burnability.

In Appendix A we list all subconfigurations of energy ≤ 12 in the sea of 4. The most important observations can be summarized as follows:

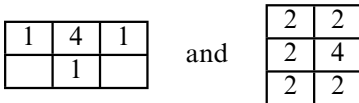
Lemma 5.1. Islands with energy ≤ 11 in the sea of 3 are in one-to-one correspondence with islands with the same energy of 4. The correspondence is given by substitution $3 \longleftrightarrow 4$.

Lemma 5.2. The same correspondence can be applied to all islands with energy =12, with two exceptions

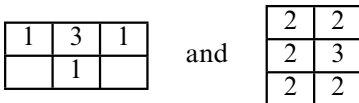


The correspondence gives all islands in the sea of 3 with energy 12.

Proof. This follows from the fact that



are burnable but



are not burnable. ■

5.1. Free Energy Estimates

Define $\Xi_{4,V}^{(12)}$ and $\Xi_{3,V}^{(12)}$ to be the partition functions of the ensembles $\mathcal{R}_{4,V}^{12}$, respectively, $\mathcal{R}_{3,V}^{12}$.

Lemma 5.3. There exists β_0 such that for any $\beta > \beta_0$ there exists a constant $c > 0$ such that for any sufficiently large volume V

$$\log \Xi_{4,V}^{(12)} - \log \Xi_{3,V}^{(12)} \geq c|V|e^{-12\beta}. \tag{5.2}$$

Proof. Define the extended ensemble $\mathcal{R}_{V,3}^{12,\text{ext}}$, consisting of *all* configurations of disjoint burnable islands with energy ≤ 12 in the sea of 3 in volume V . Evidently

$$\mathcal{R}_{V,3}^{12} \subset \mathcal{R}_{V,3}^{12,\text{ext}},$$

and the inclusion is strict because e.g.

2	3	3	2
2	3	3	2

is a subconfiguration allowed in the ensemble $\mathcal{R}_{V,3}^{12,\text{ext}}$ but not in the ensemble $\mathcal{R}_{V,3}^{12}$. Indeed, it is not burnable but the island

2
2

is burnable.

If we analogously define $\mathcal{R}_{V,4}^{12,\text{ext}}$, then in fact $\mathcal{R}_{V,4}^{12,\text{ext}} = \mathcal{R}_{V,4}^{12}$ because a configuration of (disjoint) islands in a sea of 4 is burnable if and only if all the individual islands are burnable in the sea of 4, see Proposition 2.3.

Therefore

$$\Xi_{3,V}^{(12,\text{ext})} \geq \Xi_{3,V}^{(12)},$$

and we can obtain Lemma 5.3 from the following.

Proposition 5.4.

$$\log \Xi_{4,V}^{(12)} - \log \Xi_{3,V}^{(12,\text{ext})} \geq c|V|e^{-12\beta}. \tag{5.3}$$

The proof is an application of the usual polymer expansion (see refs. 12 and 13) because islands in the sea of 4 do not interact (again from

Proposition 2.3). Comparing the expansions of $\log \Xi_{4,V}^{(12)}$ and $\log \Xi_{3,V}^{(12,\text{ext})}$ term by term, and using Lemmas 5.1 and 5.2, we obtain a difference in the terms of order $e^{-12\beta}$ and no difference in previous terms. In the same way it is easy to prove a weaker inequality valid for all finite $V \subset \mathbb{Z}^d$.

Proposition 5.5. For $\beta > \beta_0$ there exist $c > 0$, $f \in \mathbb{R}$ such that for any finite $V \subset \mathbb{Z}^d$:

$$\begin{aligned} \log \Xi_{4,V}^{(12)} - \log \Xi_{3,V}^{(12)} &\geq \log \Xi_{4,V}^{(12)} - \log \Xi_{3,V}^{(12,\text{ext})} \\ &\geq c|V|e^{-12\beta} - f|\partial V|. \end{aligned} \tag{5.4}$$

5.2. Proof of Theorem 3.2

Again, the connected components of the set of lattice edges separating different values of neighboring spins are called contours. Each contour γ will be associated with values of spins on all squares touching γ . Consider the set $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ of all those contours of a given configuration $\eta \in \mathcal{R}_{V,3}$ which have energy (length) exceeding 12, $|\gamma_i| > 12$. We call Γ the polycontour of η . The polycontour separates the volume V into a finite number of connected domains V_1, \dots, V_m . To each domain V_i is associated the value of the spin κ_i , induced by contours γ_j neighboring this domain (so $\kappa_i = \kappa$ if the spins in the inner boundary of V_i are equal to κ). If $\Gamma = \emptyset$ then there is the domain V , to which is associated $\kappa = 3$. Denote the band consisting of squares touching Γ by $[\Gamma]$. From burnability it follows that all domains V_i for which $\kappa_i = 1, 2$ are contained in $[\Gamma]$. We denote by $V^{(3)}$ the union of those V_i for which $\kappa_i = 3$, and by $V^{(4)}$ the union of those V_i for which $\kappa_i = 4$. Now ζ , the number of 3's in V , equals

$$\zeta = \zeta^{(3)} + \zeta^{(4)},$$

where $\zeta^{(3)}$, respectively, $\zeta^{(4)}$ denotes the number of 3's in $V^{(3)}$, respectively, $V^{(4)}$. Therefore

$$\text{Prob}[\zeta \geq \alpha|V|] \leq \text{Prob}\left[\zeta^{(3)} \geq \frac{\alpha}{2}|V|\right] + \text{Prob}\left[\zeta^{(4)} \geq \frac{\alpha}{2}|V|\right].$$

We will separately estimate the two terms in the right-hand side. The last one is the easiest. By definition of $V^{(4)}$, all spin values equal to 3 in $V^{(4)}$ are contained in the islands listed above. Each island contains ≤ 8 squares and has an energy ≥ 4 . So the energy per square is $\geq 1/2$. If $\zeta^{(4)} \geq |V|\alpha/2$, then the energy of the configuration is $\geq \alpha|V|/4$. Note that $\Xi_{3,V} \geq 1$ because the configuration $\equiv 3$ has zero energy. So the probability

of the configuration η is less than or equal $\exp(-\beta\alpha|V|/4)$. As the total number of configurations does not exceed $4^{|V|}$,

$$\mu_{\beta,V}^3 \left(\zeta^{(4)} \geq \frac{\alpha}{2}|V| \right) \leq 4^{|V|} \exp \left(-\beta \frac{\alpha}{4}|V| \right), \tag{5.5}$$

which, when $\alpha\beta > 4 \log 4$, tends to zero as $|V| \rightarrow \infty$.

The rest of the proof is an estimate of $\mu_{\beta,V}^3(\zeta^{(3)} \geq \alpha|V|/2)$.

Note that $\zeta^{(3)} \leq |V^{(3)}|$, and hence

$$\mu_{V,\beta}^3(\zeta^{(3)} \geq \frac{\alpha}{2}|V|) \leq \mu_{V,\beta}^3(|V^{(3)}| \geq \frac{\alpha}{2}|V|) = \frac{Z_\alpha}{\Xi_{3,V}}, \tag{5.6}$$

where Z_α is the partition function of those configurations $\eta \in \mathcal{R}_{V,3}$ for which $|V^{(3)}| \geq \alpha/2|V|$.

It is easy to see that

$$Z_\alpha \leq \sum_{\Gamma} e^{-\beta|\Gamma|} \Xi_{3,V^{(3)}}^{(12)} \Xi_{4,V^{(4)}}^{(12)}, \tag{5.7}$$

where the sum runs over those polycontours Γ , for which $|V^{(3)}| \geq \alpha/2, |V|$ ($\Gamma = \emptyset$, for which $V^{(3)} = V$ is included).

Let us now estimate $\Xi_{3,V}$. Consider only those configurations $\eta \in \mathcal{R}_{V,3}$ which also belong to $\mathcal{R}_{V',4}^{12}$, where V' is V without squares touching ∂V . Then we have

$$\Xi_{3,V} \geq e^{-3\beta|\partial V|} \Xi_{4,V'}^{(12)}. \tag{5.8}$$

From the cluster expansion (or by direct counting),

$$\Xi_{4,V'}^{(12)} \geq e^{-h|\partial V|} \Xi_{4,V}^{(12)} \tag{5.9}$$

for some constant $h > 0$. Therefore,

$$\Xi_{3,V} \geq e^{-d|\partial V|} \Xi_{4,V}^{(12)} \tag{5.10}$$

for some constant d (depending on β). So we obtain

$$\frac{Z_\alpha}{\Xi_{3,V}} \leq e^{d|\partial V|} \frac{\sum_{\Gamma} e^{-\beta|\Gamma|} \Xi_{3,V^{(3)}}^{(12)} \Xi_{4,V^{(4)}}^{(12)}}{\Xi_{4,V}^{(12)}}, \tag{5.11}$$

where the sum over Γ is as in (5.7). On the right-hand side use that for every $\xi_1 \in \mathcal{R}_{V^{(3)},4}^{12}$ and for every $\xi_2 \in \mathcal{R}_{V^{(4)},4}^{12}$, there exists a $\zeta \in \mathcal{R}_{V,4}^{12}$ that coincides with ξ_1 on $V^{(3)}$ and with ξ_2 on $V^{(4)}$. Hence,

$$\Xi_{4,V}^{(12)} \geq \Xi_{4,V^{(3)}}^{(12)} \Xi_{4,V^{(4)}}^{(12)} \tag{5.12}$$

and thus

$$\frac{Z_\alpha}{\Xi_{3,V}} \leq e^{d|\partial V|} \sum_{\Gamma} e^{-\beta|\Gamma|} \frac{\Xi_{3,V^{(3)}}^{(12)}}{\Xi_{4,V^{(3)}}^{(12)}}. \tag{5.13}$$

Now use the inequality (5.4) in order to obtain the following estimate

$$(4.6) \leq (4.13) \leq e^{d|\partial V|} \sum_{\Gamma} e^{-\beta|\Gamma|} e^{f|\partial V^{(3)}|} e^{-c|V^{(3)}|} e^{-12\beta}. \tag{5.14}$$

But evidently $|\partial V^{(3)}| \leq |\partial V| + |\Gamma|$, and $V^{(3)} \geq \alpha|V|/2$, so we have the estimate

$$e^{(d+f)|\partial V|} e^{-\frac{1}{2}\alpha c|V|} e^{-12\beta} \sum_{\Gamma} e^{-\beta'|\Gamma|}, \tag{5.15}$$

where $\beta' = \beta - f$. To estimate the last sum is standard combinatorics:

$$\sum_{\Gamma} e^{-\beta'|\Gamma|} \leq \prod_{x \in X} \left(1 + \sum_{\gamma \ni x, |\gamma| > 12} e^{-\beta'|\gamma|} \right), \tag{5.16}$$

where X is the set of all vertices—ends of edges inside V , $|X| \leq |V|$. A classical Peierls' estimate gives

$$\sum_{\gamma \ni x, |\gamma| > 12} e^{-\beta'|\gamma|} \leq H' e^{-13\beta'} \leq H e^{-13\beta} \tag{5.17}$$

for some constants $H = H' \exp[13f]$. Since

$$1 + H e^{-13\beta} \leq e^{H e^{-13\beta}},$$

we obtain the estimate

$$(4.6) \leq e^{(d+f)|\partial V|} e^{-\frac{1}{2}\alpha c|V|} e^{-12\beta} e^{H|V|} e^{-13\beta}. \tag{5.18}$$

If $\beta > \log[2H/\alpha c]$, this product tends to zero as $|V| \rightarrow \infty$, $|\partial V|/|V| \rightarrow 0$. That together with (4.5) concludes the proof.

Remark 5.6. To prove that the density of 1's or 2's goes to zero is even easier: Let N be the number of 1's and 2's in V , $N = N^{(3,4)} + N^{[\Gamma]}$ where $N^{(3,4)}$ is the number of 1's and 2's in $V^{(3)} \cup V^{(4)}$ and $N^{[\Gamma]}$ is the number of 1's and 2's in $[\Gamma]$. Observe that $N^{[\Gamma]} \leq 2|\Gamma|$, and the 1's and 2's in $V^{(3)} \cup V^{(4)}$ are contained in islands. For the islands, the energy per square is $\geq 1/2$. The energy of a configuration is equal to $|\Gamma|$ plus the energy of the islands. Hence, if $N \geq \alpha|V|$, then that energy is not less than $\alpha|V|/2$. We can then do as in (4.5) to prove that

$$\mu_{\beta,V}^3(N \geq \alpha|V|) \rightarrow 0 \quad \text{as } |V| \rightarrow +\infty$$

whenever $\alpha\beta > 2 \log 4$.

The same proof as above remains valid for the ensemble $\mathcal{R}_{V,4}$. In fact, it is valid modulo straightforward modifications, for general boundary conditions as well. The proof for the ensemble with general boundary conditions ξ only needs small modifications concerning the islands touching the boundary.

APPENDIX A: ISLANDS IN THE SEA OF 4

Here we give all subconfigurations of energy ≤ 12 in the sea of 4. Of course, these islands have to be burnable. In the figure, empty cells are part of the sea.

- (1) Energy = 4:

$$\boxed{1} \ , \ \boxed{2} \ , \ \boxed{3}$$

- (2) Energy = 6:

$$\boxed{2 \ 2} \ , \ \boxed{3 \ 3} \ \text{and rotations and reflections}$$

(this remark will be omitted from here on)

- (3) Energy = 7

$$\boxed{1 \ 2} \ , \ \boxed{1 \ 3} \ , \ \boxed{2 \ 3}$$

- (4) Energy = 8

$$\begin{array}{|c|c|} \hline a & \\ \hline & b \\ \hline \end{array} \ , \ \text{where } a, b = 1, 2, 3.$$

2	2	2
---	---	---

,

3	3	3
---	---	---

,

2	2
	2

,

3	3
	3

,

3	3
3	3

(5) Energy = 9

a	b	b
---	---	---

,

a	b
	b

 where $a = 1, 2, 3, b = 2, 3, a \neq b$.

(6) Energy = 10

a	b	c
---	---	---

,

a	b
	c

 where $a \neq b, b \neq c; a, b, c = 1, 2, 3$

and if $a = c = 1$ then $b = 3$.

a	a	
		b

,
 where $a = 2, 3, b = 1, 2, 3$.

a	a
	a
	a

,

	a
a	a
	a

,

a	a	a	a
---	---	---	---

,

a	
a	a
	a

where $a = 2, 3$.

2	2
3	2

,

3	3
2	3

,

3	3
1	3

,

	3	3
3	3	3

,

3	3	3
3	3	3

(7) Energy = 11

a	b	
		c

,
 where $a, b, c = 1, 2, 3, a \neq b$.

b	b
a	c

,
 where $b = 2, 3, a, c = 1, 2, 3, a \neq b, a \neq c, b \neq c$.

a	a	a
		b

,

a	a	a
	b	

a	a
	a
	b

,

a	a	b
	a	

a	a	
	a	b

,

a	a	a	b
---	---	---	---

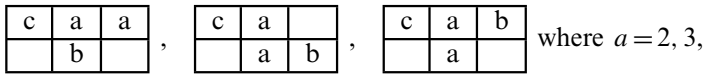
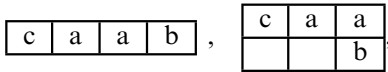
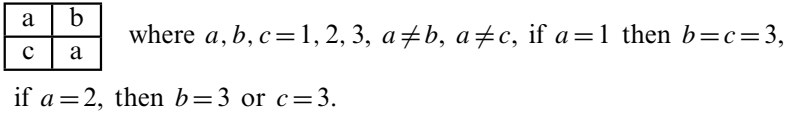
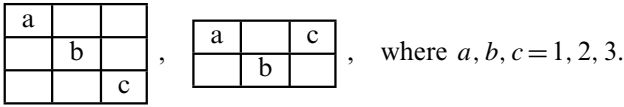
 where $a = 2, 3, b = 1, 2, 3,$

$b \neq a$.

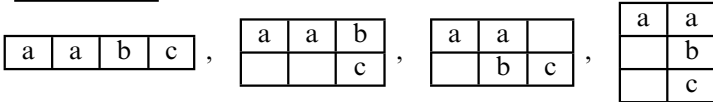
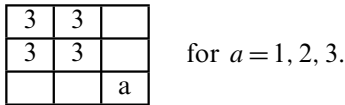
3	3	
3	3	a

,
 where $a = 1, 2$.

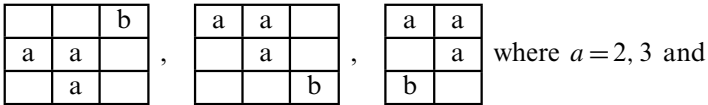
(8) Energy = 12



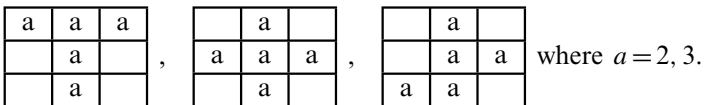
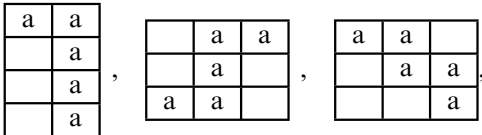
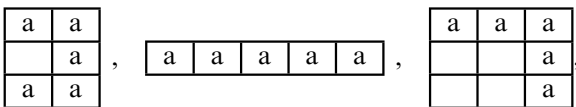
$b, c = 1, 2, 3$, $a \neq b$, $a \neq c$; if $a = 2$, then $b = 3$ or $c = 3$.

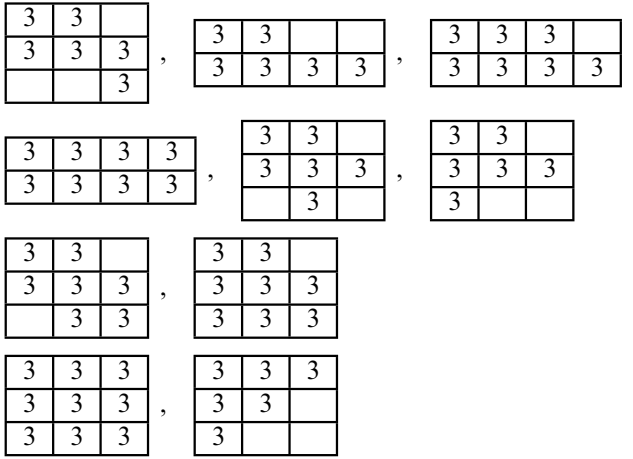


where $a, b, c = 1, 2, 3$, $a \neq 1$, $a \neq b$, $b \neq c$.



$b = 1, 2, 3$.





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